Lefschetz Coincidence Theory for Maps Between Spaces of Different Dimensions

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Abstract. For a given pair of maps $f, g : X \to M$ from an arbitrary topological space to an n-manifold, the Lefschetz homomorphism is a certain graded homomorphism $\Lambda_{fg} : H(X) \to H(M)$ of degree $(-n)$. We prove a Lefschetz-type coincidence theorem: if the Lefschetz homomorphism is nontrivial then there is an $x \in X$ such that $f(x) = g(x)$.

1. Introduction.

Consider the Fixed Point Problem: "If X is a topological space and $g : X \to X$ is a map, what can be said about the set $Fix(g)$ of points $x \in X$ such that $g(x) = x$?" The Coincidence Problem is concerned with the same question about two maps $f, g : X \to Y$ and the set $Coin(f, g)$ of $x \in X$ such that $f(x) = g(x)$.

If X is a sufficiently "nice" space (e.g., a polyhedron) then one may associate to $g : X \to X$ an integer Λ_g, called the Lefschetz number (see [**6**]):

$$\Lambda_g = L(g_*) = \sum_n (-1)^n Trace(g_{*n}),$$

where g_{*n} is the endomorphism of the n-th homology group $H_n(X)$ of X induced by f. Then the famous Lefschetz fixed point theorem states that $\lambda_g \neq 0 \Rightarrow Fix(g) \neq \emptyset$. Now suppose we are given a pair of continuous maps $f, g : X \longrightarrow Y$, where only Y has to be a "nice" space and X is arbitrary. Then one can define the Lefschetz number Λ_{fg} of the pair (f, g) (see [**3**, Section VI.14]):

$$\Lambda_{fg} = L(g_* f_!) = \sum_n (-1)^n Trace(g_{*n} f_!),$$

where $f_! : H(Y) \to H(X)$ is a certain "transfer" homomorphism of f. Then a Lefschetz-type coincidence theorem states that $\lambda_{fg} \neq 0 \Rightarrow Coin(f, g) \neq \emptyset$.

Lefschetz coincidence theory has been developed for the following settings.

Case 1: $f : (M_1, \partial M_1) \to (M_2, \partial M_2)$ is a boundary-preserving map between two n-manifolds with (possibly empty) boundaries ∂M_1 and ∂M_2, $g : M_1 \to M_2$ is arbitrary.

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For closed manifolds, this is the setting of the original Lefschetz's result. After many years, his theorem was extended to the case of manifolds with boundary by M. Nakaoka [**24**] and V. P. Davidyan [**8**], [**9**].

Case 2: $f : X \to V$ maps a topological space to an open subset of $\mathbf{R}^n$ and all fibres $f^{-1}(y)$ are acyclic, $g : X \to V$ is compact.

This approach was developed by S. Eilenberg and D. Montgomery [**12**] and later by L. Gorniewicz [**18**], A. Granas [**19**] and others. These results treat fixed points of a multivalued map $G : Y \to Y$ by letting X be the graph of G and f, g be the projections, then $Fix(G) \equiv \{x : x \in G(x)\} = Coin(f, g)$.

In [**25**] we proved a Lefschetz-type coincidence theorem that contains Cases 1 and 2 and gave examples of coincidence situations not covered by the known results (see [**25**, Section 5]). In particular we showed that the projection of the torus $\mathbf{T}^2$ on the circle $\mathbf{S}^1$ has a coincidence with any homologically trivial (inessential) map. This is an example of a map between manifolds of different dimensions. In the present paper we generalize the main results of [**25**] in order to include a Lefschetz-type coincidence theorem for the following setting.

Case 3: $f : T \times Y \to Y$ is the projection, Y is an ANR, T is any normal space, $g : T \times Y \to Y$ is arbitrary.

Here the coincidence set of the pair (f, g), $Coin(f, g)$, is the fixed point set $Fix(g) \equiv \{(t, x) : g(t, x) = x\}$ of the "parametrized" map g. This situation was studied by Knill [**22**] and later by Geoghegan and Nicas [**14, 15**], Geoghegan, Nicas and Oprea [**16**]. The Lefschetz number is replaced with a certain homomorphism $L(g) : H(T) \to H(Y)$ of degree 0 which is proven to be equal to the following.

DEFINITION 1.1. [**22, 16**] The *Knill trace* of g is defined by

$$L(g)(u) = \sum_{k \geq 0} (-1)^{k+m} \sum_{j=1}^{\beta_k} x_j^k \frown g_*(u \times b_j^k),$$

where $u \in H_m(T)$ and for each $k \geq 0$ $\{b_j^k : j = 1, ..., \beta_k\}$ is a basis for $H_k(Y)$ with corresponding dual basis $\{x_j^k : j = 1, ..., \beta_k\}$ for $H^k(Y)$. [Comment: The above definition uses Spanier's sign convention [**26**], we use Dold's sign convention [**11**] instead.]

In this paper we use the results and techniques of our previous paper [**25**] to extend some of the definitions and theorems of [**16**] to the general case of two arbitrary maps $f, g : X \to Y$, i.e., f is not necessarily the projection, with the following reservation. For the sake of simplicity, we limit our attention to the case when $Y = M$ is a manifold. Then the Lefschetz homomorphism is a certain graded homomorphism $\Lambda_{fg} : H(X, A) \to H(M)$ of degree $(-n)$, where $n = \dim M$. All relevant examples assume that X is a manifold as well and that

$$\dim X > \dim M.$$

We should tell from the start that the Lefschetz homomorphism of any pair of maps $f, g : \mathbf{S}^N \to \mathbf{S}^n$ is trivial if $N \neq n$.

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The Setup. Throughout the paper we assume the following. By H (H^*) we denote the singular (co)homology with coefficients in a field R, M is an oriented

connected compact n-manifold, $n \geq 0$, with boundary ∂M and interior $\overset{\circ}{M}$, $O_M \in H_n(M, \partial M)$ is the fundamental class of $(M, \partial M)$, X is a topological space, $A \subset X$. We will study the existence of coincidences of maps

$$f : (X, A) \to (M, \partial M),\ g : X \to M,$$

that satisfy

$$Coin(f, g) \cap A = \emptyset.$$

2. The Lefschetz Class of a Homomorphism of Degree m.

The following is a setup of the theory of the Lefschetz class of a graded homomorphism of arbitrary degree as presented in [**16**, Sections 1 and 3] (see also [**25**] and [**11**, p. 207-208]). Let E and C be graded R-spaces, E finitely generated. Let E^* denote the dual graded R-space:

$$E^q = Hom(E_q), \quad E^* = \{E^q\}, \text{ and let}$$
$$(E^* \otimes E)_m = \bigotimes\nolimits_{q-p=m} (E^p \otimes E_q), \quad E^* \otimes E = \{(E^* \otimes E)_m\}.$$

Suppose we have two graded homomorphisms

$$\frown : (E^* \otimes E)_m \longrightarrow C_m, \text{ and}$$
$$\theta : (E^* \otimes E)_m \longrightarrow Hom_m(E, E)$$

where $Hom_m(E, E)$ denotes the space of all graded homomorphisms of degree m. We define θ as follows:

$$[\theta(a \otimes b)](u) = (-1)^{|b| \cdot |u|} a(u) \cdot b,$$

where $a \in E^k, b \in E_{m+k}, u \in E_k$, $a \otimes b \in (E^* \otimes E)_m$, $|w|$ stands for the degree of w. Then by [**11**, Proposition VII.6.3, p. 208], θ is an isomorphism.

Definition 2.1. For an endomorphism $h : E \longrightarrow E$ of degree m of a finitely generated graded module E, we define the *Lefschetz class* of h by

$$L(h) = \frown \theta^{-1}(h) \in C_m.$$

The following representation of the Lefschetz class as a Knill-like trace is proven similarly to [**16**, Proposition 1.2].

Proposition 2.2. *Let $h : E \to E$ be a homomorphism of degree m. Let $\{a_1^k, ..., a_{m_k}^k\}$ be a basis for E_k and $\{x_1^k, ..., x_{m_k}^k\}$ the corresponding dual basis for E^k. Then*

$$L(h) = \sum_k (-1)^{k(k+m)} \sum_j x_j^k \frown h(a_j^k).$$

When $m = 0$, $C_m = R$ and $\frown = e$ is the evaluation map, we have the usual Lefschetz number as the alternating sum of traces, see [**11**, p. 208].

3. The Evaluation Formula for the Lefschetz Class.

For any space Y we define the following functions:
the transposition $t : Y \times Y \longrightarrow Y \times Y$ given by $t(x, y) = (y, x)$;
the diagonal map $\delta : Y \to Y \times Y$ given by $\delta(x) = (x, x)$;
the scalar multiplication $q : R \otimes H(Y) \longrightarrow H(Y)$ given by $q(r \otimes v) = r \cdot v$;
the tensor multiplication $O_M^\times : H(Y) \longrightarrow H(M, \partial M) \otimes H(Y)$ given by $O_M^\times(v) = O_M \otimes v$;

the projection $P_k : H(Y) \longrightarrow H_k(Y)$, $k \geq 0$.

Let

$$M' = M \cup C,$$

where $C = \partial M \times [0,1)$ is the collar attached to the boundary of M. Define

$$M^\times = (M \times M', M \times M' \backslash \delta(M')$$

and *the inclusions* $i : \overset{\circ}{M} \longrightarrow M,\ I : (M, \partial M) \times \overset{\circ}{M} \longrightarrow M^\times$. If $\pi : M \times M' \to M$ is the projection on the first factor then $\zeta = (M, \pi, M \times M', \delta)$ is the tangent microbundle of M [**27**, Chapter 14] and *the Thom isomorphism* $\varphi : H(M^\times) \to H(M)$ is given by $\varphi(x) = \pi_*(\tau \frown x)$, where and τ is the Thom class of ζ.

The proofs in this section follow the ideas of Dold [**10**] and [**11**, Section VI.6], see also [**18**] and [**25**].

Lemma 3.1 (Generalized Dold's Lemma). *Suppose that the map* $\Phi : H(M) \longrightarrow H(M)$ *is given as the composition of the following homomorphisms:*

$$\begin{aligned} \Phi : H(\overset{\circ}{M}) &\xrightarrow{O_M^\times} H(M, \partial M) \otimes H(\overset{\circ}{M}) \xrightarrow{\delta_* \otimes Id} H(M, \partial M) \otimes H(M) \otimes H(\overset{\circ}{M}) \\ &\xrightarrow{Id \otimes t_*} H(M, \partial M) \otimes H(\overset{\circ}{M}) \otimes H(M) \xrightarrow{I_* \otimes Id} H(M^\times) \otimes H(M) \\ &\xrightarrow{P_n \otimes Id} H_n(M^\times) \otimes H(M) \simeq R \otimes H(M) \xrightarrow{q} H(M). \end{aligned}$$

Then

$$\Phi = i_*.$$

Proof. In [**25**, Theorem 7.2] we proved this statement for the following composition:

$$\begin{aligned} \Psi : H(K) &\xrightarrow{O_K^\times} H(V, V \backslash K) \otimes H(K) \xrightarrow{\delta_* \otimes Id} H(V, V \backslash K) \otimes H(V) \otimes H(K) \\ &\xrightarrow{Id \otimes t_*} H(V, V \backslash K) \otimes H(K) \otimes H(V) \xrightarrow{I_* \otimes Id} H(M^\times) \otimes H(V) \\ &\xrightarrow{P_n \otimes Id} H_n(M^\times) \otimes H(V) \simeq R \otimes H(V) \xrightarrow{q} H(V), \end{aligned}$$

where M is a closed manifold and $(M, V, V \backslash K)$ is an excisive triad. If M has the boundary the proof is the same except for the last step: for any $p \in \overset{\circ}{M}$, $I_*(O_M \otimes p) = 1$. This formula follows from definitions of the Thom class and the fundamental class, see [**27**, Chapter 14]. Now we obtain the above statement by putting $V = M, K = \overset{\circ}{M}$. □

Now let $E = C = H(M)$ and let $\frown$ be the usual cap-product.

For the following three lemmas we fix $m \geq 0$ and let $h : H_*(M) \to H_{*+m}(M)$ be a homomorphism of degree m.

Lemma 3.2. *There is a homomorphism* $J : H_{n-*}(M, \partial M) \longrightarrow H^*(M)$ *such that the following diagram commutes*

$$\begin{array}{ccc} H_{n-*}(M, \partial M) \otimes H_{m+*}(\overset{\circ}{M}) & \xrightarrow{J \otimes i_*} & H^*(M) \otimes H_{m+*}(M) \\ \downarrow^{I_*} & & \downarrow^{\frown} \\ H_{n+m}(M^\times) & \xrightarrow{\varphi} & H_m(M), \end{array}$$

or

$$\frown (J \otimes i_*) = \varphi I_*.$$

Proof. Let J be given by

$$J(u)(v) = \varphi I_*(u \otimes v), \quad u \in H_{n-j}(M, \partial M), v \in H_{m+j}(M).$$

Let's consider the diagram on the chain level and start in the left upper corner with $u \otimes v$, where u is a $(n-j)$-chain and v is a $(m+j)$-chain. Then, in terms of the Alexander-Whitney approximation, we get $\pi_* \tau I_*(u \otimes_j \lfloor v) v \rfloor_m$ in the right lower corner. □

Lemma 3.3. *Let*

$$a = (J \otimes h)\delta_*(O_M) \in (H^*(M) \otimes H(M))_m.$$

Then

$$\frown (a) = \varphi I_*(Id \otimes i_*^{-1} h)\delta_*(O_M) \in H_m(M).$$

Proof. By Lemma 3.2 we have

$$\frown (a) = \frown (J \otimes i_*)(Id \otimes i_*^{-1} h)\delta_*(O_M) = \varphi I_*(Id \otimes i_*^{-1} h)\delta_*(O_M).$$

□

Lemma 3.4. *The following diagram commutes:*

$$\begin{array}{ccc}
H_{n-k}(M, \partial M) \otimes H_k(M) \otimes H_p(\mathring{M}) & \xrightarrow{J \otimes h \otimes i_*} & H^k(M) \otimes H_{m+k}(M) \otimes H_p(M) \\
\downarrow{\scriptstyle Id \otimes t_*} & & \downarrow{\scriptstyle Id \otimes t_*} \\
H_{n-k}(M, \partial M) \otimes H_p(\mathring{M}) \otimes H_k(M) & \xrightarrow{J \otimes i_* \otimes h} & H^k(M) \otimes H_p(M) \otimes H_{m+k}(M) \\
\downarrow{\scriptstyle I_* \otimes Id} & & \downarrow{\scriptstyle \frown \otimes Id} \\
H_{n-k+p}(M^\times) \otimes H_k(M) & \xrightarrow{\varphi \otimes h} & H_{p-k}(M) \otimes H_{m+k}(M) \\
\downarrow{\scriptstyle q(P_n \otimes Id)} & & \downarrow{\scriptstyle q(P_0 \otimes Id)} \\
H_k(M) & \xrightarrow{h} & H_{m+k}(M).
\end{array}$$

Proof. The first square trivially commutes. For the second, going ↓→, we get $\varphi I_* \otimes h$. Now going →↓, from Lemma 3.2 we get:

$$(\frown \otimes Id)(J \otimes i_* \otimes h) = (\frown (J \otimes i_*) \otimes h) = \varphi I_* \otimes h.$$

The third square commutes as $p = k$. □

Theorem 3.5 (Evaluation Formula). *For any homomorphism $h : H(M) \to H(M)$ (of degree m) we have*

$$L(h) = \varphi I_*(Id \otimes i_*^{-1} h)\delta_*(O_M).$$

Proof. (cf. [**25**, Theorem 8.5]) We start in the left upper corner of the above diagram with $\delta_*(O_M) \otimes v$, where $v \in H(M)$. Let $u = i_*(v)$. Then going ↓→, we get $h(u)$ by Lemma 3.1. Let $a = \sum_i a_{i-m} \otimes a'_i$ with $a_{i-m} \in H^{i-m}(M), a'_i \in H_i(M)$.

Then going $\to\downarrow$, we get:

$$\begin{aligned}
&q(P_0\otimes Id)(\frown\otimes Id)(Id\otimes t_*)(J\otimes h\otimes Id)(\delta_*(O_M)\otimes u)\\
&=q(P_0\frown\otimes Id)(Id\otimes t_*)((J\otimes h)\delta_*(O_M)\otimes u)\\
&=q(P_0\frown\otimes Id)(Id\otimes t_*)(a\otimes u)\qquad\text{by definition of } a \text{ (Lemma 3.3)}\\
&=q(P_0\frown\otimes Id)(Id\otimes t_*)(\textstyle\sum_i a_{i-m}\otimes a_i'\otimes u)\\
&=q(P_0\frown\otimes Id)\textstyle\sum_i(-1)^{|a_i'|\cdot|u|}(a_{i-m}\otimes u\otimes a_i')\\
&=\textstyle\sum_i(-1)^{|a_i'|\cdot|u|}P_0(a_{i-m}\frown u)\cdot a_i'\\
&=\textstyle\sum_i(-1)^{|a_i'|\cdot|u|}a_{i-m}(u)\cdot a_i'\\
&=\textstyle\sum_i\theta(a_{i-m}\otimes a_i')(u)\\
&=\theta(a)(u).
\end{aligned}$$

Thus $\theta(a)=h:H(M)\longrightarrow H(M)$. Therefore by definition of the Lefschetz homomorphism we have

$$L(h)=L(\theta(a))=\frown(a).$$

Now the statement follows from Lemma 3.3. □

4. Transfers and the Coincidence Homomorphism.

Since $Coin(f,g)\cap A=\emptyset$, the map $(f\times g)\delta:(X,A)\longrightarrow M^\times$ is well defined.

Definition 4.1. The *coincidence homomorphism* I_{fg} *of the pair* (f,g) is the homomorphism $I_{fg}:H(X,A)\to H(M^\times)$ of degree 0 defined by

$$I_{fg}=(f\times g)_*\delta_*.$$

It is clear that $I_{fg}\neq 0\Longrightarrow Coin(f,g)\neq\emptyset$.

From dimensional considerations we obtain the following.

Proposition 4.2. *If* $z\in H_i(X,A)$*,* $i<n$ *or* $i>2n$*, then* $I_{fg}(z)=0$.

This is the reason why it is often sufficient to consider *the coincidence index with respect to* $\mu\in H_n(X,A)$, as in [**25**]:

$$I_{fg}(\mu)=(f\times g)_*\delta_*(\mu).$$

In fact when X is a manifold and $\deg X=n=\deg M$, the best we can do is to consider $I_{fg}(O_X)$, where O_X is the fundamental class of X (see also Corollaries 6.6 and 6.7).

Corollary 4.3. I_{fg} *is trivial for maps* $\mathbf{S}^N\to M$ *if* $N>2n$.

Definition 4.4. The *transfer (or the shriek map) of* f *with respect to* $z\in H_{n+m}(X,A)$ is the homomorphism $f_!^z:H_*(M)\longrightarrow H_{*+m}(X)$ of degree m given by

$$f_!^z=(f^*D^{-1})\frown z,$$

where $D:H^*(M,\partial M)\to H_{n-*}(M)$ is the Poincare-Lefschetz duality isomorphism $D(x)=x\frown O_M$.

When X is a manifold and z is its fundamental class, $f_!^z=f_!$ is the usual transfer homomorphism, or an *Umkehr*-homomorphism, of f [**11**, Section VIII.10], or a shriek map [**3**, p. 368].

We can represent the coincidence homomorphism via the transfer as follows.

THEOREM 4.5 (Representation Formula).

$$I_{fg}(z) = I_*(Id \otimes i_*^{-1} g_* f_!^z)\delta_*(O_M).$$

PROOF. In [**25**, Theorem 2.1] the formula is proven for $z \in H_n(X, A)$, but the proof is valid for any $z \in H(X, A)$. $\square$

Thus $I_{fg}(z)$ is the image of O_M under the composition of the following maps:

$$H(M, \partial M) \xrightarrow{\delta_*} H(M, \partial M) \otimes H(M) \xrightarrow{Id \otimes i_*^{-1} h} H(M, \partial M) \otimes H(\overset{\circ}{M}) \xrightarrow{I_*} H(M^{\times}),$$

where homomorphism $h = g_* f_!^z : H(M) \to H(M)$ of degree m is defined by the following diagram:

$$\begin{tikzcd} H^*(X, A) \arrow[d, "\frown z"] & H^*(M, \partial M) \arrow[l, "f^*"'] \arrow[d, "D"] \\ H(X) \arrow[r, "g_*"] & H(M). \end{tikzcd}$$

For a given map $f : X \to Y$, Gottlieb [**20**] (see also [**1**] and [**21**]) defines *a partial transfer of f with trace k* as a homomorphism $\tau : H(Y) \to H(X)$ such that $f_* \tau : H(Y) \to H(Y)$ is the multiplication by k:

$$f_* \tau = k \cdot Id.$$

Then independently of Theorem 4.5 we can prove its analogue (for $m = 0$):

PROPOSITION 4.6. *Suppose τ is a partial transfer of $f : X \to M$ ($\partial M = \emptyset$) with trace $k \neq 0$. Then*

$$I_{fg}(z) = I_*(Id \otimes g_* \tau)\delta_*(O_M),$$

where $z = \tau(O_M)$.

PROOF. Consider the following commutative diagram:

$$\begin{tikzcd} & H(M) \arrow[r, "\delta_*"] & H(M) \otimes H(M) & \\ H(M) \arrow[ur, "k \cdot Id"] \arrow[r, "\tau"] & H(X) \arrow[u, "f_*"] \arrow[r, "\delta_*"] & H(X) \otimes H(X) \arrow[u, "f_* \otimes f_*"] & H(M) \otimes H(M) \arrow[l, "\tau \otimes \tau"'] \arrow[ul, "k^2 \cdot Id"'] \end{tikzcd}$$

Then going from the left to the right we get $\frac{1}{k}\delta_*$. Hence $\delta_* \tau = (\tau \otimes \tau)\frac{1}{k}\delta_*$, therefore $k \cdot \delta_* \tau = (\tau \otimes \tau)\delta_*$. Thus the next diagram is commutative:

$$\begin{tikzcd} H(M) \arrow[r, "\tau"] \arrow[d, "\delta_*"] & H(X) \arrow[d, "k \cdot \delta_*"] \\ H(M) \otimes H(M) \arrow[r, "\tau \otimes \tau"] \arrow[dr, "k \cdot Id \otimes \tau"] & H(X) \otimes H(X) \arrow[d, "f_* \otimes Id"] \\ & H(M) \otimes H(X). \end{tikzcd}$$

Then

$$(Id \otimes \tau)\delta_* = (f_* \otimes Id)\delta_* \tau,$$

so

$$I_*(Id \otimes g_* \tau)\delta_* = I_*(f_* \otimes g_*)\delta_* \tau.$$

Now the statement follows from the definition of I_{fg}. $\square$

The transfer $f_!^z$ of f defined above can be an example of a partial transfer (see [**20**, Proposition 1] or [**3**, Proposition VI.14.1 (6), p. 394]):

Proposition 4.7. *If there is a* $z \in H(X,A)$ *such that* $f_*(z) = k \cdot O_M$ *then* $f_!^z$ *is a partial transfer of* f *with trace* k.

Thus any partial transfer with nonzero trace satisfies the statement of Theorem 4.5 and, therefore, the Lefschetz-type Theorem 6.1 below. On the other hand, $k \neq 0$ is a strong restriction as it implies that $f_*(z) = k \cdot O_M$, which in case of manifolds of equal dimensions means that $\deg f \neq 0$.

5. The Lefschetz Homomorphism of the Pair.

For a fixed $z \in H_{n+m}(X,A)$, the homomorphism $g_* f_!^z : H(M) \to H(M)$ has degree m. Then by Definition 2.1 we have $L(g_* f_!^z) \in H_m(M)$.

Definition 5.1. The *Lefschetz homomorphism* $\Lambda_{fg} : H_*(X,A) \to H_{*-n}(M)$ *of the pair* (f,g) is the homomorphism of degree $(-n)$ given by

$$\Lambda_{fg}(z) = L(g_* f_!^z),\ \ z \in H(X,A).$$

Remark 5.2. Suppose $N > n$. Then from dimensional considerations it follows that $\Lambda_{fg} = 0$ for maps $f, g : \mathbf{S}^N \to \mathbf{S}^n$.

Remark 5.3. Suppose $z \in H_i(M, \partial M)$ and $i > n$. Then $z = 0$, therefore $\Lambda_{Id,Id}(z) = 0$.

Remark 5.4. Since the degree of $g_* f_!^z$ is $|z| - n$, the Lefschetz homomorphism is represented as a Knill-like trace (Proposition 2.2):

$$\Lambda_{fg}(z) = \sum_k (-1)^{k(k+|z|-n)} \sum_j x_j^k \frown g_* f_!^z(a_j^k),$$

where $\{a_1^k, ..., a_{m_k}^k\}$ is a basis for $H_k(M)$ and $\{x_1^k, ..., x_{m_k}^k\}$ the corresponding dual basis for $H^k(M)$.

The Lefschetz homomorphism satisfies a naturality property below. The formula is similar to the one in [**16**, Theorem 2.6] but the proof is much shorter because we do not use the Knill trace (see also Theorem 6.2).

Theorem 5.5 (Naturality I). *Let* (Y,B) *be a topological space and* $h : (Y,B) \to (X,A)$ *be a map. Then*

$$\Lambda_{fh,gh} = \Lambda_{fg} h_*.$$

Proof. Let $z \in H(Y,B)$. Then we have

$$(gh)_*(fh)_!^z = g_* h_*(h^* f^* D^{-1} \frown z) = g_*(f^* D^{-1} \frown h_*(z)) = g_* f_!^{h_*(z)}.$$

□

Corollary 5.6. *If* h *has a partial transfer* τ *with trace* k *then*

$$\Lambda_{fh,gh}\tau = k \cdot \Lambda_{fg}.$$

Propositions 5.5 and 5.6 generalize the well known formula for maps between two n-manifolds [**3**, Corollary VI.14.6, p. 297]:

$$L(fh, gh) = \deg(h) L(f,g),$$

where $L(f,g)$ is the ordinary Lefschetz number.

The following corollary shows how the Lefschetz homomorphism generalizes the Knill trace of a parametrized map defined in [**16**, Definition 2.1], see also Corollary 6.4.

COROLLARY 5.7. *Suppose* $(X,A) = Y \times (M, \partial M)$, $g : X \to M$ *is a map and* $p : Y \times (M, \partial M) \to (M, \partial M)$ *is the projection (then* $Fix(g) = Coin(p, g)$*). Then*

$$\Lambda_{pg}(u \times O_M) = L(g_{u*}),\ \ u \in H(Y),$$

where $g_u : H(M) \to H(M)$ *is given by* $g_u(x) = (-1)^{(n-|x|)|u|} g_*(u \times x)$.

PROOF. Let $x \in H(M)$ and suppose $x = z \frown O_M$ for some $z \in H^*(M)$. Then we have

$$\begin{aligned} p^*(z) \frown (u \times O_M) &= (1 \times z) \frown (u \times O_M) = (-1)^{|z||u|}(1 \frown u) \times (z \frown O_M) \\ &= (-1)^{(n-|x|)|u|} u \times x. \end{aligned}$$

Therefore

$$g_* p_!^{u \times O_M}(x) = g_*(p^*(z) \frown (u \times O_M)) = (-1)^{(n-|x|)|u|} g_*(u \times x)$$

and the statement follows. $\square$

6. Further Properties.

Theorems 3.5 and 4.5 imply the following theorem.

THEOREM 6.1 (Lefschetz-Type Coincidence Theorem). *The coincidence homomorphism is equal to the Lefschetz homomorphism:*

$$\varphi I_{fg} = \Lambda_{fg}.$$

Moreover, if $\Lambda_{fg} \neq 0$, *then* (f, g) *has a coincidence.*

According to Corollary 5.7, Λ_{pg} generalizes the Knill trace of a parametrized map $g : Y \times M \to M$, while in [**16**] the Knill trace is defined for $F : Y \times (X, A) \to (X, A)$, i.e., as a map of pairs. But the result corresponding to the one above is due to Knill [**22**, Theorem 1] and is proven for the case of $F : Y \times X \to X$.

The above identity allows us to establish some facts about the Lefschetz homomorphism that are hard to obtain directly from its definition.

THEOREM 6.2 (Naturality II). *(cf.* [**16**, Theorem 2.6]*) Let* (X', A') *be a topological space,* $(M', \partial M')$ *another* n*-manifold,* $h : (X, A) \to (X', A')$, $k : (M, \partial M) \to (M', \partial M')$, $f', g' : (X', A') \to (M', \partial M')$ *maps, and* $f'h = kf$, $g'h = kg$, *i.e., we have the following two (in one) commutative diagrams:*

$$\begin{array}{ccc} (X,A) & \xrightarrow{\ \ f,g\ \ } & (M,\partial M) \\ \downarrow^{h} & & \downarrow^{k} \\ (X',A') & \xrightarrow{\ \ f',g'\ \ } & (M',\partial M'). \end{array}$$

Suppose also that k *is a homeomorphism. Then*

$$k_* \Lambda_{fg} = \Lambda_{f'g'} h_*.$$

PROOF. The fact that k is a homeomorphism implies two things. First, $k \times k : M^{\times} \to (M')^{\times}$ is well defined. Hence from the naturality of the Thom isomorphism we have

$$k_* \varphi = \varphi'(k \times k)_*,$$

where φ' is the Thom isomorphism for M'. Second, since $Coin(f,g) \cap A = \emptyset$, it follows that $Coin(kf,kg) \cap A = \emptyset$ and, therefore, $I_{kf,kg}$ is well defined (Definition 4.1). In the computation below we also use Theorem 5.5 (a similar statement can be proven independently for I_{fg}), Theorem 6.1 and the trivial fact that $I_{kf,kg} = (k \times k)_* I_{fg}$. We have

$$\Lambda_{f'g'} h_* = \Lambda_{f'h,g'h} = \Lambda_{kf,kg} = \varphi' I_{kf,kg} = \varphi'(k \times k)_* I_{fg} = k_* \varphi I_{fg} = k_* \Lambda_{fg}.$$

Observe that the Lefschetz homomorphism Λ_{fg} is well defined without the restriction $Coin(f,g) \cap A = \emptyset$. $\square$

Even when $A = \partial M = \emptyset$, the definition of the Lefschetz homomorphism is not symmetric, but the one of the coincidence homomorphism (Definition 4.1) is, as follows:

$$I_{fg}(z) = t_* I_{gf}(z).$$

Now we use the fact that $t_*(x) = (-1)^n x$ for $x \in H(M^\times)$ (the proof of this formula is dual to the proof of Lemma 6.16 of [**28**, p. 165]). As a result we have the following property.

Proposition 6.3 (Symmetry). *Suppose $f, g : X \to M$ are maps ($\partial M = \emptyset$). Then*

$$\Lambda_{fg}(z) = (-1)^n \Lambda_{gf}(z), \ z \in H(X).$$

It follows that $\Lambda_{ff} = 0$ when n is odd (in particular, $\chi(M) = 0$).

Now we can obtain another representation of the Knill trace of a parametrized map, in terms of the Lefschetz homomorphism (cf. Corollary 5.7):

Corollary 6.4. *Suppose $g : Y \times M \to M$ ($\partial M = \emptyset$) is a map and $p : Y \times M \to M$ is the projection. Then*

$$L(g_{u*}) = (-1)^n \Lambda_{gp}(u \times O_M), \ u \in H(Y),$$

where $g_u : H(M) \to H(M)$ is given by $g_u(x) = (-1)^{(n-|x|)|u|} g_*(u \times x)$.

Proof. It follows from Corollary 5.7 and Proposition 6.3. $\square$

The proof of following property of the coincidence homomorphism is trivial.

Theorem 6.5 (Product Theorem). *(cf. [**16**, Theorem 4.5]) Let (X', A') be a topological space, $(M', \partial M')$ a manifold, $f', g' : (X', A') \to (M', \partial M')$ maps. Then there is a commutative diagram:*

$$\begin{array}{ccc} H((X,A) \times (X',A')) & \xrightarrow{I_{f\times f', g\times g'}} & H((M \times M')^\times) \\ \uparrow \xi & & \uparrow C_*\eta \\ H(X,A) \otimes H(X',A') & \xrightarrow{I_{fg} \otimes I_{f'g'}} & H(M^\times) \otimes H((M')^\times), \end{array}$$

where $C : M^\times \times (M')^\times \to (M \times M')^\times$ is the map which interchanges the middle factors, ξ and η are the Künneth isomorphisms.

Under certain circumstances the Lefschetz homomorphism is trivial in all dimensions but n.

Corollary 6.6. *If $z \in H_i(X, A)$, $i \neq n$, then $\Lambda_{ff}(z) = 0$.*

Proof. If $i > n$ then $\Lambda_{ff}(z) = \Lambda_{Id,Id} f_*(z) = 0$ by Proposition 5.5 and Remark 5.3. The rest follows from Proposition 4.2. $\square$

Corollary 6.7. *Suppose $g_* = 0$ in reduced homology. If $z \in H_i(X,A)$, $i \neq n$, then $\Lambda_{fg}(z) = 0$.*

Proof. It follows from Proposition 4.2. □

Thus, when g is homologically trivial it suffices to consider only the Lefschetz class with respect to an element $z \in H_n(X,A)$, see [**25**, Section 5]. In fact the following condition is sufficient for Λ_{fg} to be nontrivial (cf. Proposition 4.7):

(A): $f_* : H_n(X,A) \to H_n(M,\partial M)$ is a nonzero homomorphism.

Proposition 6.8. [**25**, Corollary 5.1] *If f satisfies condition (A) and $g_* = 0$ in reduced homology then $\Lambda_{fg} \neq 0$.*

We call a map $f : (X,A) \to (M,\partial M)$ *weakly coincidence-producing* if every map $g : X \to M$ with $g_* = 0$ has a coincidence with f (compare to coincidence producing maps [**7**, Section 7]). Now we can restate Proposition 6.8.

Corollary 6.9. *If f satisfies condition (A) then f is weakly coincidence-producing.*

Let's consider some examples of applications of this corollary.

Corollary 6.10. [**25**, Corollary 5.6] *Suppose M is a homotopy sphere, $f : X \to M$ is a map, and*

(A′): *$f_\# : \pi_n(X) \to \pi_n(M)$ is onto.*

Then condition (A) is satisfied, so f is weakly coincidence-producing.

The proposition below follows from Lemma 5 of Gottlieb [**20**].

Corollary 6.11. *Let M, X be smooth closed manifolds, $f : X \to M$ be smooth, $\dim X = N, N > n$. Suppose $F = f^{-1}(y)$ is a fiber with $y \in M$ a regular point (then F is a closed $(N-n)$-manifold) and*

(A″): *$i^* : H^{N-n}(X) \to H^{N-n}(F)$ is nonzero.*

Then condition (A) is satisfied, so f is weakly coincidence-producing.

Corollary 6.12. *Let $f : X \to M$ be an orientable fibration with fiber F. Suppose F is arcwise connected, $A = f^{-1}(\partial M)$, $\partial M \neq \emptyset, H_i(F) = 0$ for $0 < i < n-1$. Then condition (A) is satisfied, so f is weakly coincidence-producing.*

Proof. Suppose $\mathcal{C}$, a Serre class, contain only the zero group, see [**26**, Theorem 9.6.10, p. 506]. Then $\mathcal{C}$ is an ideal of abelian groups. Observe that $H_i(M,\partial M) \in \mathcal{C}$ for $0 \leq i < 1$, and $H_j(F) \in \mathcal{C}$ for $0 < j < n$. Then $f_* : H_q(X,A) \to H_q(M,\partial M)$ is an $\mathcal{C}$-epimorphism for $q \leq n$, so (A) is satisfied. □

7. Examples.

Brown [**5**] proved that a compact closed manifold M is suitable (see [**13**]) if and only if there is a multiplication on M such that:

(1) $x \cdot e = x,\ \forall x \in M$;

(2) $\forall a, b \in M,\ \exists x \in M$ such that $a \cdot x = b$;

(3) $x \cdot y = x \cdot z \Rightarrow y = z,\ \forall x, y, z \in M$.

Then M is an H-manifold and for any $x \in M$ there is a unique $x^{-1} \in M$ such that $x \cdot x^{-1} = e$. The following is a slight generalization of Theorem 3 of Wong [**29**].

THEOREM 7.1. *If M is a suitable manifold, $A=\varnothing$, then*

$$\Lambda_{fg}(z) = <\overline{O_M}, \psi_*(z)>, \ z \in H_n(X),$$

where $\psi(x) = g(x) \cdot [f(x)]^{-1}$, $x \in X$, and $\overline{O_M}$ is the dual of the fundamental class O_M.

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
X \times X & \xrightarrow{f \times g} & M \times M & \xrightarrow{I} & M^{\times} & \xleftarrow{j} & M, \\
\uparrow^{\delta} & & \downarrow^{\sigma} & & \downarrow^{\sigma} & \swarrow_{k} & \\
X & \xrightarrow{\psi} & M & \xrightarrow{k} & (M, M\backslash\{e\}) & &
\end{array}$$

where $\sigma(a,b) = b \cdot a^{-1}$, $j(y) = (e,y)$, k is the inclusion. Since $k_* : H_n(M) \to H_n(M, M\backslash\{e\})$ is an isomorphism, for any $z \in H_n(X)$ we have the following

$$I_{fg}(z) = I_*(f \times g)_* \delta_*(z) = j_* \psi_*(z).$$

Therefore

$$\Lambda_{fg}(z) = \varphi j_* \psi_*(z) = <\overline{O_M}, \psi_*(z)> .$$

$\square$

Corollaries 6.9 - 6.12 and Theorem 7.1 can be used to prove the existence of coincidences of maps between manifolds of different dimensions. However we do not use the whole Lefschetz homomorphism, only its part in dimension $n = \dim M$, i.e., $\Lambda_{fg} : H_n(X,A) \to H_0(M)$ (in other words, we need only the Lefschetz number with respect to a $z \in H_n(X,A)$ as in [**25**]). This is also true for Example 2.4 in [**16**]: if $A : \mathbf{S}^3 \times \mathbf{S}^2 \to \mathbf{S}^2$ is the action given by regarding $\mathbf{S}^2$ as the homogeneous space $\mathbf{S}^3/\mathbf{S}^1$ arising from the Hopf principal bundle $\mathbf{S}^1 \to \mathbf{S}^3 \to \mathbf{S}^2$, then the Knill trace is 0 in all dimensions except 0. This means that the only nonzero part of Λ_{pA} (p is the projection) is the following: $H_2(\mathbf{S}^3 \times \mathbf{S}^2) \to H_0(\mathbf{S}^2)$.

To show that other values of the Lefschetz homomorphism may be important consider Examples 2.3 and 7.2 in [**16**]. In the notation of the present paper, the first one states the following. If $\mu : G \times G \to G$ is the multiplication of a compact Lie group then for $u \in H_{n+m}(G)$ we have

$$\Lambda_{p\mu}(u \times O_G) = \begin{cases} 0 & \text{if } m < n \\ (-1)^n u & \text{if } m = n. \end{cases}$$

The second example implies that if Y is a manifold with $\dim Y = k > 0$ and $\chi(Y) \neq 0$ then for a certain map $g : X = \mathbf{S}^m \times \mathbf{S}^m \times Y \to M = \mathbf{S}^m \times Y$, the Lefschetz homomorphism Λ_{pg} is nontrivial in dimension $m+n$ ($n = \dim M = m+k$): $H_{m+n}(X) \to H_m(M)$. Thus we have an example of the Lefschetz homomorphism with nontrivial values in dimensions other than 0 or n.

Note that the statement in [**16**] is that the maps A, μ, and g above and all maps homotopic to them have coincidences with the corresponding projections p. We have proved a little more than that: these maps have coincidences with all maps homotopic to p. To take this even further from the setting of parametrized maps, we can consider the composition of the above maps with another map. For example, suppose H is a topological space, $k : H \to G$ is a map. Let $f = p(k \times k)$, $g = \mu(k \times k) : H \times H \to G$, so neither is the projection. Then by Theorem 5.5 we have for $u \in H_{n+m}(H \times H)$:

$$\Lambda_{fg}(u \times k_*^{-1}(O_G)) = \Lambda_{p\mu}(k_*(u) \times O_G) = \begin{cases} 0 & \text{if } m < n \\ (-1)^n k_*(u) & \text{if } m = n. \end{cases}$$

In view of the Knill-like trace representation of the Lefschetz homomorphism (Remark 5.4) we obtain the following.

PROPOSITION 7.2. *For $z \in H_{2n}(X, A)$, we have $\Lambda_{fg}(z) = g_*(f^*(\overline{O_M}) \frown z)$.*

It follows that if X is a compact orientable $2n$-manifold, $f^{*n} \neq 0$ and $\ker g_{*n} = 0$ then $\Lambda_{fg}(O_X) \neq 0$, where O_X is the fundamental class of X.

Final Remarks.

(1) The statement of Theorem 3.5 holds for open subsets of $\mathbf{R}^q$, see [**25**, Sections 6-10]. As a result, the second part of Theorem 6.1 can be proven for spaces more general than manifolds (such as ANR's) by following Gorniewicz [**18**, Sections V.2 and V.3]. It would also be interesting to try to obtain Theorem 6.1 for a non-orientable manifold M by following Gonçalves and Jezierski [**17**].

(2) We know that the Hopf map $h : \mathbf{S}^3 \to \mathbf{S}^2$ is onto, in other words, it has a coincidence with any constant map c. On the other hand, as $\Lambda_{fg} = 0$ for any pair of maps $f, g : \mathbf{S}^N \to \mathbf{S}^n$ and $N \neq n$, the Lefschetz-type Coincidence Theorem 6.1 fails to predict the existence of coincidences of (h, c). In fact, h has a coincidence with any map homotopic to c [**4**], therefore the converse of the Lefschetz coincidence theorem for spaces of different dimensions fails.

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